

On a class of generating vector fields for the extremum seeking problem: Lie bracket approximation and stability properties*

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Abstract

In this paper, we describe a broad class of control functions for extremum seeking problems. We show that it unifies and generalizes the existing extremum seeking strategies which are based on Lie bracket approximations, and allows to design new controls with favorable properties in extremum seeking and vibrational stabilization tasks. The second result of this paper is a novel approach for studying the asymptotic behavior of solutions to extremum seeking systems. It provides a constructive procedure for defining frequencies of control functions to ensure the practical asymptotic and exponential stability. In contrast to many known results, we also prove asymptotic stability in the sense of Lyapunov for the proposed class of extremum seeking systems under appropriate assumptions on their vector fields.

1 Introduction

In many control applications, the goal is to operate a system in some optimal fashion. Often, however, the optimal operating point is unknown or may even change over time so that it cannot be determined a priori. Extremum seeking control is a control methodology to solve such problems of stabilizing and tracking an a priori unknown optimal operating point. Typically, it is model-free and minimizes or maximizes the steady-state map of a system. The steady-state map maps constant control input values to the steady-state output values. It is a well-defined map under appropriate assumptions on the system. There exist many ways to design the extremum seeking strategies. A classical perturbation-based approach is to use the controls consisting of time-periodic oscillating inputs (often called dither, excitation, perturbation or learning signal) and state-dependent vector fields in order to gather information about the unknown steady-state map. Based on the perturbed input and the perturbed output response, typically the gradient or other descent directions of the steady-state map are approximated or estimated by appropriate signal processing or filtering methods, see, e.g. [2, 3, 4, 7, 8, 9, 10, 11, 14, 20]. Hereby, the shape of control functions plays an important role since it influences the speed of convergence and may be subject to input constraints. In the literature, different types of excitation signals have been analyzed, see, e.g. [1, 13, 17, 18, 21].

In this paper, we propose a novel class of vector fields for extremum seeking controls based on Lie bracket approximation techniques [2, 3]. The first contribution of this paper is

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the formula describing a whole class of vector fields for an extremum seeking system which allows to approximate a gradient flow in various ways. The formula unifies and generalizes previously known controls presented in [2, 17, 18, 19] and allows to generate new extremum seeking strategies with desirable properties. In particular, we demonstrate benefits of this formula by designing a control which has bounded update rates *and* vanishing amplitudes at the same time.

Moreover, the second contribution is a rigorous proof of the asymptotic stability *in the sense of Lyapunov*, under appropriate assumptions on the considered class of generating vector fields. This is in contrast to many results in the literature, where typically *practical* stability results are established. The proof extends the techniques developed in [5, 6, 22, 23]. A further advantage of these techniques is the possibility to estimate the decay rate of solutions of the extremum seeking systems.

Finally, we demonstrate that the proposed formula is not only of use in the extremum seeking but also in vibrational stabilization problems [12, 16].

This paper is organized as follows. Section 2 contains some preliminary results on the extremum seeking problem based on Lie bracket approximations. In Section 3, we present a novel formula to approximate the gradient flows and establish various asymptotic stability conditions. In Section 4, we compare several extremum seeking strategies by using numerical simulations, and illustrates the application of the obtained results to the vibration stabilization problem. Section 5 contains auxiliary lemmas and proofs of the main results. The proofs of the auxiliary results are in the Appendix.

2 Preliminaries

2.1 Notations

In this paper, \mathbb{R}^+ denotes the set of all non-negative real numbers, $B_\delta(x^*)$ is the δ -neighborhood of $x^* \in \mathbb{R}^n$, and $\overline{B_\delta(x^*)}$ is its closure. The symbol δ_{ij} denotes the Kronecker delta. For $h \in C^1(\mathbb{R}^n; \mathbb{R})$, $\xi \in \mathbb{R}^n$, we denote the vector $\frac{\partial h(x)}{\partial x}$ evaluated at $x = \xi$ by $\frac{\partial h(\xi)}{\partial x}$, and define $\nabla h(\xi) := \frac{\partial h(\xi)}{\partial x}^T$. For an integrable function $f : \mathbb{R} \rightarrow \mathbb{R}$, $\int f(z)ds$ means one of the antiderivatives of f ; $f(z) = O(z)$ as $z \rightarrow z^0$ means that there is a $c > 0$ such that $|f(z)| \leq c|z|$ in some neighborhood of z^0 . For $a, b \in \mathbb{R}^n$, we denote their open convex hull as $\text{co}\{a, b\} = \{\lambda a + (1 - \lambda)b \mid \lambda \in (0, 1)\}$.

2.2 Lie bracket approximations

Consider a control-affine system

$$\dot{x} = f_0(x) + \sum_{j=1}^l f_j(x) \sqrt{\omega} u_j(\omega t), \quad (1)$$

where $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$, $x(t_0) = x_0 \in \mathbb{R}^n$, $t_0 \in [0, \infty)$, $\omega > 0$, $f_i \in C^2(\mathbb{R}^n; \mathbb{R}^n)$, $i = 0, 1, \dots, l$, $u_j \in L^\infty(\mathbb{R}^+; \mathbb{R})$ are T -periodic with some $T > 0$, and $\int_0^T u_j(\tau) d\tau = 0$, $j = \overline{1, l}$.

It can be shown that the trajectories of (1) approximate trajectories of the following *Lie bracket system*:

$$\dot{\bar{x}} = f_0(\bar{x}) + \frac{1}{T} \sum_{i < j} [f_i, f_j](\bar{x}) \int_0^T \int_0^\theta u_j(\tau) u_i(\tau) d\tau d\theta, \quad (2)$$

where $[f_i, f_j](\bar{x}) = \frac{\partial f_j(\bar{x})}{\partial x} f_i(\bar{x}) - \frac{\partial f_i(\bar{x})}{\partial x} f_j(\bar{x})$, $\bar{x}(t_0) = x^0$, see [2] and references therein. The stability properties of systems (1) and (2) are related as follows.

Lemma 1 ([2]). *If a compact set $S \subset \mathbb{R}^n$ is locally (globally) uniformly asymptotically stable for (2) then it is locally (semi-globally) practically uniformly asymptotically stable for (1).*

Below we recall the notion of practical stability.

Definition 1. *A compact set $S \subset \mathbb{R}^n$ is said to be locally practically uniformly asymptotically stable for (1) if:*

- *it is practically uniformly stable, i.e. for every $\varepsilon > 0$ there exist $\delta > 0$ and $\omega_0 > 0$ such that, for all $t_0 \geq 0$ and $\omega > \omega_0$, if $x^0 \in B_\delta(S)$ then the corresponding solution of (1) satisfies $x(t) \in B_\varepsilon(S)$ for all $t \geq t_0$;*
- *$\hat{\delta}$ -practically uniformly attractive with some $\hat{\delta} > 0$, i.e. for every $\varepsilon > 0$ there exist $t_1 \geq 0$ and $\omega_0 > 0$ such that, for all $t_0 \geq 0$ and $\omega > \omega_0$, if $x^0 \in B_{\hat{\delta}}(S)$ then the corresponding solution of (1) satisfies $x(t) \in B_\varepsilon(S)$ for all $t \geq t_0 + t_1$;*
- *the solutions of system (1) are practically uniformly bounded, i.e. for every $\delta > 0$ there exist $\varepsilon > 0$ and $\omega_0 > 0$ such that for all $t_0 \geq 0$ and $\omega > \omega_0$, if $x^0 \in B_\delta(S)$ then $x(t) \in B_\varepsilon(S)$ for all $t \geq t_0$.*

If the attractivity property holds for every $\hat{\delta} > 0$, then the set S is called semi-globally practically uniformly asymptotically stable for (1).

2.3 Extremum seeking problem

In this paper, we address a class of extremum seeking problems related to the unconstrained minimization of a continuously differentiable cost function J . We assume that $J \in C^1(\mathbb{R}^n; \mathbb{R})$ is unknown (as an analytic expression) but can be evaluated (measured) at each $x \in \mathbb{R}^n$. The goal is to construct a control system of the form $\dot{x} = u(t, J(x))$ such that the (local) minima of J have some desired stability properties for this system. In this simple setup, the (static) map J corresponds to the steady-state map of a system. However, the extremum seeking based on Lie bracket approximations can be applied to much more general scenarios, including dynamic maps (dynamical systems), constrained optimization problems, distributed and multi-agent extremum seeking, stabilization, synchronization and consensus problems as well as problems on manifolds, etc. The results obtained in this paper can be applied to such more general problems but are not discussed here for the sake of simplicity.

The underlying idea of the extremum seeking based on the Lie bracket approximations is as follows. Suppose that $n = 1$, i.e. $x \in \mathbb{R}$, and consider the system

$$\dot{x} = 2J(x)\sqrt{\omega}\cos(\omega t) + \sqrt{\omega}\sin(\omega t). \quad (3)$$

It can be seen that the Lie bracket system for (3) approximates the gradient flow of J :

$$\dot{\bar{x}} = [J(\bar{x}), 1] = -\nabla J(\bar{x}). \quad (4)$$

Thus, the trajectories of system (3) approximate trajectories of the gradient flow of J and they converge, for example, if J is convex, into an arbitrary small neighborhood of the set of minima of J , for ω sufficiently large. For $n > 1$, the gradient flow can be approximated in a similar way, see [2] for details.

3 Main results

3.1 Vector fields for approximating gradient flows

Observe that there are many ways to define the vector fields of system (1) such that the corresponding Lie bracket system has the form (4). For example, consider the system

$$\dot{x} = \frac{1}{2}e^{J(x)}\sqrt{\omega}\cos(\omega t) + e^{-J(x)}\sqrt{\omega}\sin(\omega t). \quad (5)$$

Computing $[\frac{1}{2}e^{J(x)}, e^{-J(x)}]$ yields $-\nabla J(x)$ and, hence, the associated Lie bracket system is again of the form (4). The main idea and the first main result of this paper is the description of a class of vector fields for system (1) such that the corresponding Lie bracket system (2) represents a gradient-like flow of J .

Consider first the system

$$\dot{x} = F_1(J(x))\sqrt{\omega}u_1(\omega t) + F_2(J(x))\sqrt{\omega}u_2(\omega t), \quad (6)$$

where $x \in \mathbb{R}$, $\omega > 0$. We begin with the one-dimensional case $x \in \mathbb{R}$ to simplify the presentation, and the multi-dimensional case will be considered later as an extension.

Theorem 1. *Let $F_1, F_2 \in C^1(\mathbb{R}; \mathbb{R})$, $F_1(z) \neq 0$, satisfy*

$$F_2(z) = F_1(z) \int \frac{F_0(z)}{F_1(z)^2} dz, \quad (7)$$

with some $F_0 : \mathbb{R} \rightarrow \mathbb{R}$, and let $u_s : \mathbb{R}^+ \rightarrow \mathbb{R}$ be T -periodic, $\int_0^T u_s(\tau) d\tau = 0$, $s=1, 2$, $\int_0^T \int_0^p u_2(\theta)u_1(\tau) d\tau d\theta = \beta T$, with $\beta > 0$. Then the Lie bracket system for (6) has the form

$$\dot{\bar{x}} = -\beta \nabla J(\bar{x}) F_0(J(\bar{x})). \quad (8)$$

Proof. Consider the differential equation

$$F_1(z) \frac{dF_2(z)}{dz} - F_2(z) \frac{dF_1(z)}{dz} = F_0(z), \quad z \in \mathbb{R}, \quad (9)$$

and observe that it can be represented as

$$F_1(z)^2 \frac{d}{dz} \left(\frac{F_2(z)}{F_1(z)} \right) = F_0(z)$$

if $F_1(z) \neq 0$. Therefore, the class of functions satisfying (9) is given by formula (7) under the assumption that the right-hand side of (7) is well-defined. Then, by computing the Lie bracket system for (6), we obtain (8):

$$\begin{aligned} \dot{\bar{x}} &= \frac{1}{T} [F_1(J(\bar{x})), F_2(J(\bar{x}))] \int_0^T \int_0^\theta u_2(\theta)u_1(\tau) d\tau d\theta \\ &= \beta \left(F_2(J(\bar{x})) \frac{dF_1(J(\bar{x}))}{dJ} - F_1(J(\bar{x})) \frac{dF_2(J(\bar{x}))}{dJ} \right) \nabla J(\bar{x}) \\ &= -\beta \nabla J(\bar{x}) F_0(J(\bar{x})). \end{aligned}$$

□

Remark 1. Formula (7) with $F_0(J(x)) = \text{const}$ describes a whole class of vector fields F_1, F_2 such that the trajectories of system (6) approximate trajectories of the gradient flow (8). Moreover, formula (7) unifies and generalizes some known results which are discussed in the following. In [2], the case $F_0(z) = 1$,

$$F_1(z) = 2z, \quad F_2(z) = 1$$

was considered, which corresponds to system (3). In [17], the case $F_0(z) = 1$,

$$F_1(z) = \sin(z), \quad F_2(z) = \cos(z)$$

was considered, which has a useful property to have a priori bounds on the vector fields (i.e. one has bounded update rates) for a fixed ω , due to the property $-1 \leq F_s(z) \leq 1$, $s = 1, 2$. In [19], the case $F_0(z) = 1$

$$F_1(z) = \sqrt{z}\sin(\ln(z)), \quad F_2(z) = \sqrt{z}\cos(\ln(z)) \quad (10)$$

was considered for $z \geq 0$, which has a nice feature that the vector fields of (6) tend to zero whenever $z = J(x)$ approaches the origin and when zero is the minimal value of J . It should be noted that in (10) J has to be smooth enough to ensure the continuous differentiability of the vector fields of system (6). This formula is useful when the minimal value of $J(x^*)$ is known a priori, but not the extremum point x^* itself. For example, such situations arise in the distance minimization, consensus or synchronization problems and, as we will see in the next section, in vibrational stabilization problems where J plays the role of a Lyapunov function.

Besides unifying these known results, formula (7) allows also to construct novel controls with desirable properties. In particular, it is possible to combine the advantage of having bounded update rates and vanishing perturbation amplitudes, e.g., this can be achieved with $F_0(z) = 1$ and controls with desirable properties. For example, it is possible to combine the advantage of having bounded update rates and vanishing perturbation amplitudes, this can be achieved, e.g., with $F_0(z) = 1$ and

$$F_1(z) = \sqrt{\frac{1 - e^{-z}}{1 + e^z}} \sin(e^z + 2 \ln(e^z - 1)),$$

$$F_2(z) = \sqrt{\frac{1 - e^{-z}}{1 + e^z}} \cos(e^z + 2 \ln(e^z - 1)),$$

for $z > 0$, $F_1(0) = F_2(0) = 0$. For $z \geq 0$, one has $|F_s(z)| \leq \sqrt{3 - 2\sqrt{2}}$, $s = 1, 2$.

Formula (7) can also be used to approximate gradient-like flows of multivariable cost functions. Consider the system

$$\dot{x} = \sum_{i=1}^n \left(F_{1i}(J(x))u_{1i}(t) + F_{2i}(J(x))u_{2i}(t) \right) e_i, \quad (11)$$

where $x \in \mathbb{R}^n$, $J \in C^2(\mathbb{R}^n; \mathbb{R})$, and e_i denotes the i -th unit vector in \mathbb{R}^n .

Theorem 2. Suppose that each pair $F_{1i}, F_{2i} : \mathbb{R} \rightarrow \mathbb{R}$ satisfies relation (7) with $F_{0i} : \mathbb{R} \rightarrow \mathbb{R}$. Define $u_{si}(t) := \sqrt{\omega} \tilde{u}_{si}(\omega t)$, $s = 1, 2$, $i = \overline{1, n}$, $\omega > 0$, where the functions \tilde{u}_{si} are T -periodic and satisfy $\int_0^T \tilde{u}_{si}(\tau) d\tau = 0$,

$$\int_0^T \int_0^\theta \tilde{u}_{2i}(\theta) \tilde{u}_{1i}(\tau) d\tau ds = \beta_i T \quad \text{with some } \beta_i > 0,$$

$$\int_0^T \int_0^\theta \tilde{u}_{si}(\theta) \tilde{u}_{pj}(\tau) d\tau d\theta = \delta_{ij} (1 - \delta_{sp}), \quad p=1, 2, j=\overline{1, n}.$$

Then the Lie bracket system for (11) has the form

$$\dot{\bar{x}} = - \sum_{i=1}^n \beta_i \frac{\partial J(\bar{x})}{\partial \bar{x}_i} F_{0i}(J(\bar{x})) e_i. \quad (12)$$

Moreover, if $F_{1i}(J(\cdot)), F_{2i}(J(\cdot)) \in C^2(\mathbb{R}^n; \mathbb{R})$, and a compact set $S \subset \mathbb{R}^n$ is locally (globally) uniformly asymptotically stable for (12), then S is locally (semi-globally) practically uniformly asymptotically stable for (11).

The proof follows directly from Lemma 1 and Theorem 1.

3.2 Stability conditions

In this section, we establish the second main result of the paper. Namely, for certain functions J , we prove the practical *exponential* stability of the extremum point for system (11). It is important that, unlike many existing results in the extremum seeking literature, we also present conditions for *asymptotic stability in the sense of Lyapunov*.

A. Practical asymptotic and exponential stability

We will refer to the following assumption.

Assumption 1. The cost function J and the vector fields F_{si} in (11) satisfy the following conditions:

A1 There exists an $x^* \in \mathbb{R}^n$ such that $\nabla J(x^*) = 0$ and $\nabla J(x) \neq 0$ for all $x \in D \setminus \{x^*\} \subset \mathbb{R}^n$, where $D = B_\Delta(x^*)$ with some $\Delta > 0$.

A2 $J(x^*) = J^*$ and $J(x) > J^*$ for all $x \in D \setminus \{x^*\}$, i.e. x^* is an isolated local minimum, and the attained minimal value of J at x^* is J^* .

A3 There exist constants $\gamma_1, \gamma_2, \kappa_1, \kappa_2, \mu$, and $m_1 \geq 1$, such that, for all $x \in D$,

$$\begin{aligned} \gamma_1 \|x - x^*\|^{2m_1} &\leq J(x) - J^* \leq \gamma_2 \|x - x^*\|^{2m_1}, \\ \kappa_1 (J(x) - J^*)^{2 - \frac{1}{m_1}} &\leq \|\nabla J(x)\|^2 \leq \kappa_2 (J(x) - J^*)^{2 - \frac{1}{m_1}}, \\ \left\| \frac{\partial^2 J(x)}{\partial x^2} \right\| &\leq \mu (J(x) - J^*)^{1 - \frac{1}{m_1}}. \end{aligned}$$

A4 There exists an $L > 0$ such that $|F_{si}(J(x) - J^*) - F_{si}(J(y) - J^*)| \leq L \|x - y\|$ for all $x, y \in D$, $s=1, 2$, $i=\overline{1, n}$.

Without loss of generality, we assume $t_0 = 0$. We will use the following trigonometric inputs in (11) (however, some other inputs are possible, see, e.g. [21]):

$$\begin{aligned} u_{1i}(t) &= u_{1i}^\varepsilon(t) = 2\sqrt{\frac{\pi k_i}{\varepsilon}} \cos\left(\frac{2\pi k_i t}{\varepsilon}\right), \\ u_{2i}(t) &= u_{2i}^\varepsilon(t) = 2\sqrt{\frac{\pi k_i}{\varepsilon}} \sin\left(\frac{2\pi k_i t}{\varepsilon}\right), \end{aligned} \quad (13)$$

where $k_i \in \mathbb{N}$, $k_i \neq k_j$ for all $i \neq j$, and ε is a positive parameter. Note that such inputs satisfy the assumptions of Theorem 2 with $\omega = \varepsilon^{-1}$, $T = \varepsilon$, $\beta_i = 1$. We underline the dependence of controls on ε by using the superscript u_{si}^ε in (13). The proofs of the theorems proposed in this section can be found in Section 5.

We claim the *practical asymptotic and exponential stability* of x^* for system (11) assuming $F_0(J) = 1$ in (7). Although practical asymptotic stability can be proven with other methods (see Theorem 2), our result allows to estimate the decay rate of solutions, and its proof presents a constructive procedure for defining ε in (13).

Theorem 3. Let F_{1i}, F_{2i} in (11) satisfy relation (7) with $F_{0i}(J)=1$, $F_{si}(J(\cdot)-J^*) \in C(D; \mathbb{R}) \cap C^2(D \setminus \{x^*\}; \mathbb{R})$ for each $i=\overline{1, n}$, $s = 1, 2$, and let Assumption 1 holds.

Then, for each $\rho > 0$, there exist $\delta > 0$, $\hat{\varepsilon} > 0$ such that, for any $\varepsilon \in (0, \hat{\varepsilon})$, the solutions of system (11) with $x^0 \in B_\delta(x^*)$ and $u_{1i}^\varepsilon(t), u_{2i}^\varepsilon(t)$ defined by (13), $i = \overline{1, n}$, satisfy

$$\|x(t) - x^*\| \leq \varphi(\|x^0 - x^*\|, t, \varepsilon) + \psi_\varepsilon, \text{ for all } t \in [0, t_1],$$

with some $t_1 \geq 0$, and

$$\|x(t) - x^*\| \leq \rho \text{ for all } t \in [t_1, \infty).$$

Here, $\psi_\varepsilon \in (0, \min\{\varepsilon, \rho\})$, $\psi_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, $\varphi(\|x^0 - x^*\|, t, \varepsilon) = \sigma \|x^0 - x^*\| e^{-\lambda t}$ for $m_1 = 1$, $\varphi(\|x^0 - x^*\|, t, \varepsilon) = (\sigma_1 \|x^0 - x^*\|^{2(1-m_1)} - \varepsilon + \sigma_2 t)^{\frac{1}{2(1-m_1)}}$ for $m_1 > 1$, with some $\lambda, \sigma, \sigma_1, \sigma_2 > 0$.

The proof of Theorem 3 is in Section 5.

Remark 2. Note that the practical asymptotic and exponential stability conditions do not require the knowledge of x^* and $J(x^*)$, but just the local behavior of J in a neighborhood of x^* . However, as it will be shown in Theorem 4, if the minimal value of the cost function (but not x^* itself) is supposed to be known for the vector fields in (11), then the asymptotic stability in the sense of Lyapunov can be ensured. The knowledge of J^* may seem quite restrictive in the context of extremum seeking, however, as discussed in Remark 1 and Section 4, such cases are still of relevance in applications.

B. Asymptotic stability in the sense of Lyapunov

To prove the asymptotic stability in the sense of Lyapunov, we require an additional assumption on the vector fields F_{si} of system (11). As it was noted in Remark 2, the value J^* has to be known so that F_{si} can be chosen appropriately. Without loss of generality, assume $J^* = 0$.

Assumption 2. There exist $\alpha_1, \alpha_2, M, \tilde{L}, m_2 \geq 0$ such that, for all $x \in D$, $s = 1, 2$, $i = \overline{1, n}$,

$$\begin{aligned} \alpha_1 J^{m_2}(x) &\leq F_{i0}(J(x)) \leq \alpha_2 J^{m_2}(x), \\ |F_{si}(J(x))| &\leq M \|x - x^*\|^{m_1(1+m_2)}, \\ \|\nabla F_{si}(J(x))\| &\leq \tilde{L} \|x - x^*\|^{m_1(1+m_2)-1}, \end{aligned}$$

where m_1 is defined in Assumption 1.

Theorem 4. Let F_{1i}, F_{2i}, F_{0i} in (11) satisfy relation (7), $F_{si}(J(\cdot)) \in C^2(D; \mathbb{R})$, for each $i=\overline{1, n}$, $s=1, 2$, and let Assumptions 1 and 2 hold with $m_1 > 1$ or $m_2 > 0$, and $J^* = 0$.

Then there exist $\delta > 0$, $\hat{\varepsilon} > 0$, such that, for any $\varepsilon \in (0, \hat{\varepsilon})$, the solutions of system (11) with $x^0 \in B_\delta(x^*)$ and $u_{1i}^\varepsilon(t), u_{2i}^\varepsilon(t)$ defined by (13), $i = \overline{1, n}$, satisfy

$$\|x(t) - x^*\| = O\left(t^{-\frac{1}{2(m_1(1+m_2)-1)}}\right) \text{ as } t \rightarrow +\infty. \quad (14)$$

The proof is in Section 5.

Remark 3. As it is stated in Theorem 4, a special but conceptually interesting case arises if Assumption 2 holds with $m_1 > 1$ or $m_2 > 0$, since in such situations the asymptotic stability of x^* in the sense of Lyapunov is possible. The proof of the asymptotic stability

result requires subtle and non-trivial decay rate estimates of the cost functions along the solutions of system (11) with controls (13) (similarly to the approach of [5, 6]).

With the proposed approach, the “classical” exponential stability of x^* can be reached assuming $m_1=1$, $m_2=0$, $F_{si}(J(\cdot)) \in C^2(D; \mathbb{R})$ (see [22, 23]). The difficulty is that the second derivative of functions $F_{si}(J(x))$ generated by (7) with $F_{0i}(J)=1$ is unbounded in a neighborhood of x^* under Assumptions 1,2 with $m_1=1, m_2=0$. For a simple explanation of this fact, assume $x \in \mathbb{R}$, $x^*=0$. Then straightforward computations show that the second derivative of $F_2(J(x)) = F_1(J) \int \frac{dJ}{F_1(J)^2} \Big|_{J=J(x)}$ contains, in particular, the term $\zeta = F_1^{-1}(J(x)) \frac{d^2 J(x)}{dx^2}$. Under Assumptions 1,2 with $m_1=1, m_2=0$, and $\inf_x \left| \frac{d^2 J(x)}{dx^2} \right| > 0$, we may conclude that there exists a $c > 0$ such that $|\zeta| \geq c|x|^{-1}$ in $D \setminus \{0\}$.

Thus, we may guarantee only the exponential stability of an arbitrary small neighborhood of x^* , similarly to Theorem 3. However, in this case the behavior of solutions of the extremum seeking systems is, in general, better than under the conditions of Theorem 3: indeed, if Assumption 2 holds with $m_1 = 1$, $m_2 = 0$ then $|F_{si}(J(x))| \leq M\|x - x^*\|$, so that $F_{si}(J(x))$ tend to 0 as $J(x) \rightarrow J^*$. The property $F_{si}(J(\cdot)) \in C^2(D; \mathbb{R})$ is possible if $m_1 > 1$ or $m_2 > 0$, as it is formulated in Theorem 4. This will be illustrated with an example in the next section.

4 Examples

4.1 Extremum seeking

As it has already been mentioned, formula (7) describes a whole class of vector fields in (6) with various properties for approximating gradient-like flows of the cost function. In this section, we illustrate the behavior of solutions of (11) with different vector fields discussed in Section 3 and controls of type (13). For numerical simulation, in each example we choose the cost function $J_1(x) = (x - x^*)^2$, $x \in \mathbb{R}$, $x^*=1$, $J^*=0$, $k_1 = 1$. The extremum seeking system introduced in [2] is useful in practical implementations due to its simple form:

$$\dot{x} = J_1(x)u_1^\varepsilon(t) + u_2^\varepsilon(t). \quad (15)$$

It can be used for minimizing the cost functions of general form, without any information about its analytical expression and extremum values. The same property holds for the control strategy with so called bounded updated rates proposed in [17]:

$$\dot{x} = \sin(J_1(x))u_1^\varepsilon(t) + \cos(J_1(x))u_2^\varepsilon(t). \quad (16)$$

It is easy to see that both of the above strategies do not vanish at the optimization point which leads to an oscillating behavior (see Fig. 1). For problems with known value of the extremum (but not the extremum point), it is possible to achieve vanishing or small oscillations as $x(t) \rightarrow x^*$, as it is stated in Theorem 4. In particular, the following extremum seeking control strategy proposed in [19] ensures the exponential convergence to an arbitrary small neighborhood of x^* , as it is discussed in Remark 3:

$$\dot{x} = \sqrt{J_1(x)} \sin(\ln(J_1(x))) u_1^\varepsilon(t) + \sqrt{J_1(x)} \cos(\ln(J_1(x))) u_2^\varepsilon(t), \quad (17)$$

for $J_1(x) > 0$, and $\dot{x} = 0$ for $J_1(x) = 0$. Indeed, in this case, $|F_s(J_1(x))| \leq \|x - x^*\|$, $|\nabla F_s(J_1(x))| \leq 1 + \sqrt{2}$ for all $x \in \mathbb{R}$, and $\left| \frac{d^2 F_s(J_1(x))}{dx^2} \right| \leq 2\sqrt{2}\tilde{\rho}^{-1}$ for all $x : |x| \geq \tilde{\rho}$, $\tilde{\rho} > 0$.

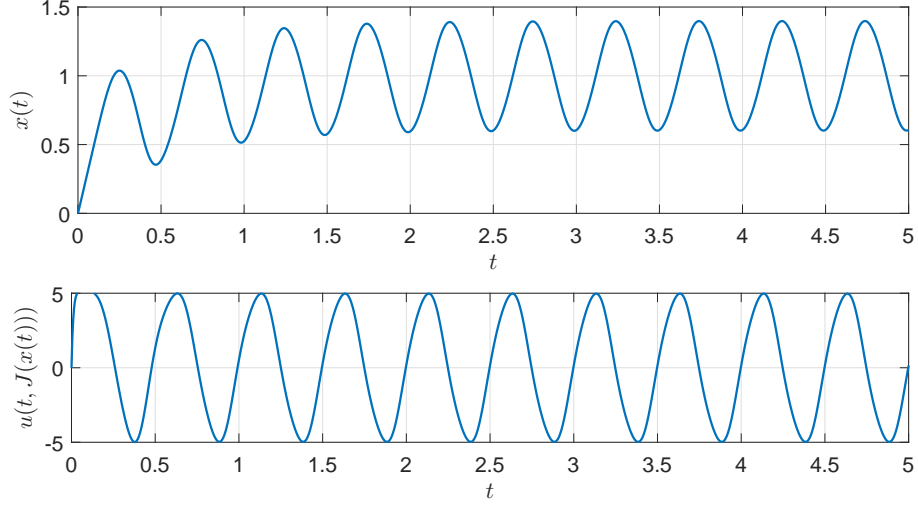


Figure 1: Trajectories (top) and controls (bottom) of system (15) with J_1 ; $\varepsilon = 0.5$.

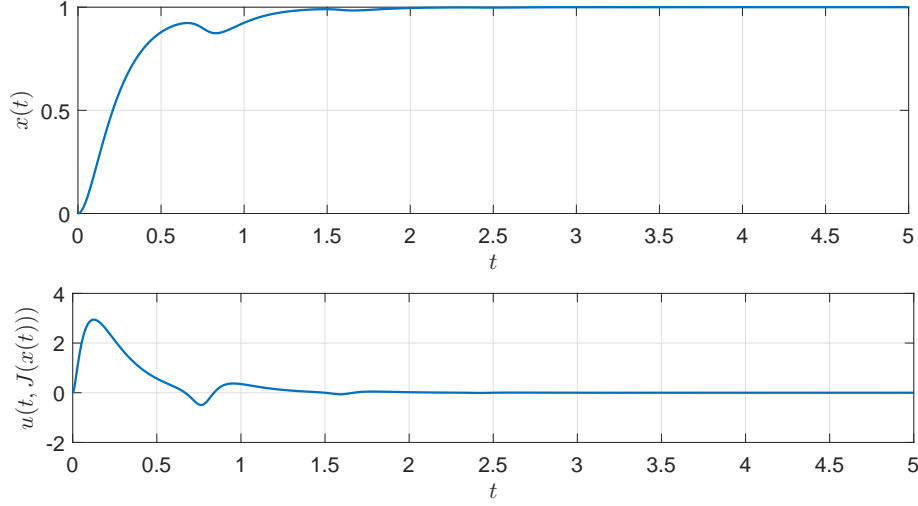


Figure 2: Trajectory (top) and control (bottom) of system (17) with J_1 ; $\varepsilon = 0.5$.

Thus, the conditions of Theorem 3 are satisfied, and, moreover, Assumption 2 holds with $m_1=1, m_2=0$.

In order to have also bounded update rates, we propose the following extremum seeking system:

$$\begin{aligned} \dot{x} = & \sqrt{\frac{1-e^{-J_1(x)}}{1+e^{J_1(x)}}} \sin(e^{J_1(x)} + 2 \ln(e^{J_1(x)} - 1)) u_1^\varepsilon(t) \\ & + \sqrt{\frac{1-e^{-J_1(x)}}{1+e^{J_1(x)}}} \cos(e^{J_1(x)} + 2 \ln(e^{J_1(x)} - 1)) u_2^\varepsilon(t), \end{aligned} \quad (18)$$

for $J_1(x) > 0$, and $\dot{x} = 0$ for $J_1(x) = 0$. Similarly to the previous example, its vector fields satisfy the assumptions of Theorem 3 (and Assumption 2 with $m_1 = 1, m_2 = 0$). Figs. 2 and 3 illustrate the behavior of trajectories of systems (17) and (18), respectively.

Note that the second derivatives of the corresponding vector fields F_1, F_2 in (17), (18) are unbounded in each neighborhood of x^* . However, for some other cost functions, e.g., $J_2(x) = (x-1)^4$, we have $F_1(J_2(\cdot)), F_2(J_2(\cdot)) \in C^2(\mathbb{R}; \mathbb{R})$. Fig. 4 shows the trajectory

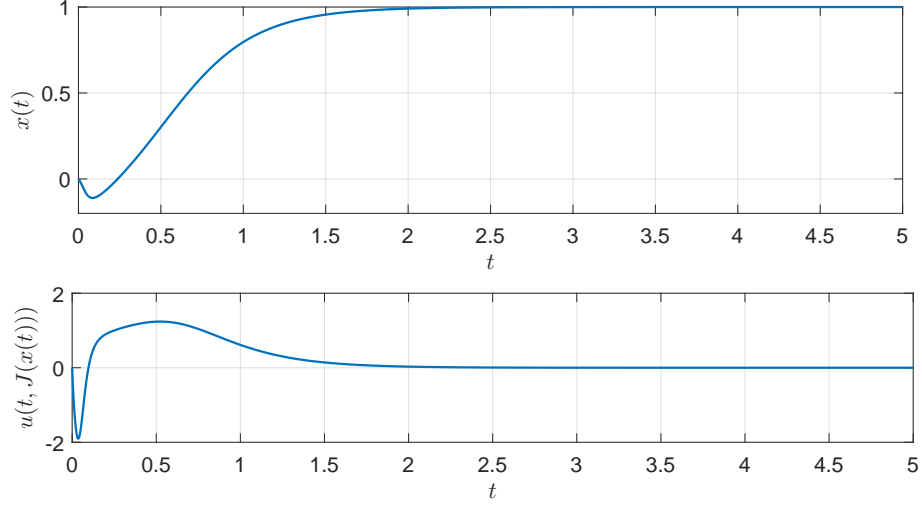


Figure 3: Trajectory (top) and control (bottom) of system (18) with J_1 ; $\varepsilon = 0.5$.

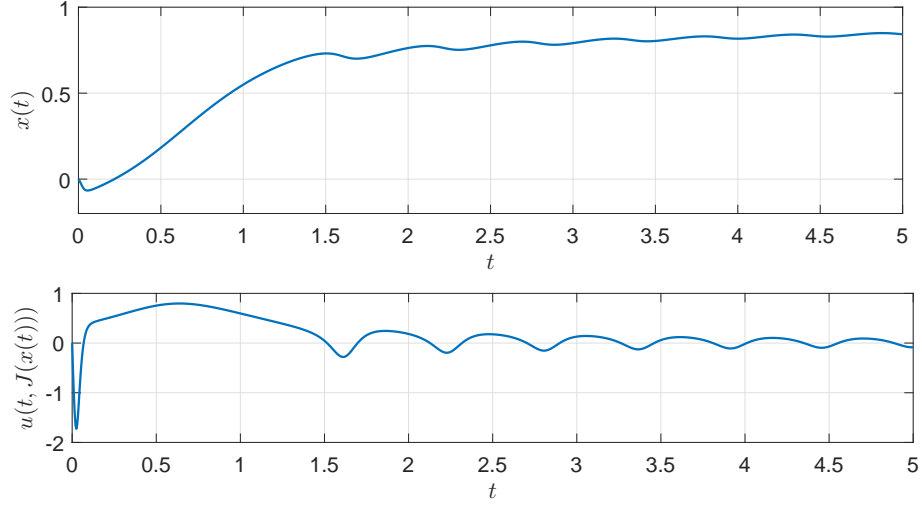


Figure 4: Trajectory (top) and control (bottom) of systems (18) with J_2 ; $\varepsilon = 0.5$.

of system (18) with J_2 . Observe that the higher order nonlinearity of J_2 results in a slower convergence of the extremum seeking algorithm in comparison with the quadratic J_1 (see Fig. 3 and Fig. 4). For J_1 , the C^2 -property can be reached by defining F_1, F_2 from formula (7) with, e.g., $F_0(J_1(x)) = J_1(x)$. For example, vector fields can be taken as $F_1(J_1(x)) = J_1(x) \sin(\ln(J_1(x)))$, $F_2(J_1(x)) = J_1(x) \cos(\ln(J_1(x)))$ for $J_1(x) > 0$. In both cases, the conditions of Theorem 4 are satisfied: for J_2 , with $m_1 = 2$, $m_2 = 0$; for J_1 , with $m_1 = m_2 = 1$.

4.2 Vibrational stabilization

Another application, where the formula (7) is of use, is found in the area of the vibrational stabilization of systems with partially or completely unknown dynamics. Consider the system

$$\dot{x} = f(x) + g(x)u, \quad (19)$$

where $x \in \mathbb{R}^n$, $f, g \in C^2(\mathbb{R}^n; \mathbb{R}^n)$, $u \in \mathbb{R}$. It was shown in [12, 16] that, under appropriate assumptions, system (19) can be practically stabilized by using the control law

$$u = V(x)\sqrt{\omega}\cos(\omega t) + 2\alpha\sqrt{\omega}\sin(\omega t), \quad (20)$$

where α is a positive constant, and V is a control Lyapunov function for (19). To see why this is possible, compute the corresponding Lie bracket system which takes the form

$$\dot{\bar{x}} = f(\bar{x}) - \alpha g(\bar{x})L_g V(\bar{x}), \quad (21)$$

where $L_g V = \nabla V^T g$. Hence, (20) approximates the control law $u_{L_g V}(x) = -\alpha L_g V(x)$, which is sometimes called damping- or $L_g V$ -control law. An interesting feature of the “vibrational” control law (20) is that it only relies on the values of the control Lyapunov function, and neither the vector field g nor the gradient of V is needed to implement this control law. Similarly to Section 3, we can construct more general control laws of the form

$$u = F_1(V(x))\sqrt{\omega}u_1(\omega t) + 2\alpha F_2(V(x))\sqrt{\omega}u_2(\omega t), \quad (22)$$

where F_1, F_2 , satisfy relation (7) with $F_0(z) = \alpha$, and u_1, u_2 satisfy the assumptions made in Theorem 1. It is easy to verify that the corresponding Lie bracket system of (22) coincides with (21), so that formula (7) allows to define a class of vibrational control laws which approximate the $L_g V$ -control laws and stabilize nonlinear systems of the form (19) without knowing f, g and the gradient of the control Lyapunov function V . Notice that the control Lyapunov functions are positive definite and hence predestinated to apply formulas with bounded update rate and vanishing amplitudes as discussed in Section 3. For a simple illustration, consider the equation

$$\dot{x} = x + \mu u, \quad (23)$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ is unknown parameter. For example, let $|\mu| \geq 1$, and take the control Lyapunov function $V(x) = x^2$, and $u_{L_g V}(x) = -2\alpha\mu x$, $\alpha > 0.5$. The evolution of the solution of system (23) with the control law (20) and the initial condition $x(0) = 1$ is presented on Fig. 5 (top). For comparison, we also take the control law of the type (18), i.e.

$$\begin{aligned} u = & \sqrt{\frac{1-e^{-V(x)}}{1+e^{V(x)}}} \sin(e^{V(x)} + 2\ln(e^{V(x)}-1))\sqrt{\omega}\cos(\omega t) \\ & + \alpha \sqrt{\frac{1-e^{-V(x)}}{1+e^{V(x)}}} \cos(e^{V(x)} + 2\ln(e^{V(x)}-1))\sqrt{\omega}\sin(\omega t), \end{aligned} \quad (24)$$

for $V(x) > 0$, and $u = 0$ for $V(x) = 0$. The corresponding trajectory is plotted on Fig. 5 (bottom). Again, we see that the vector fields that vanish at the origin show a good performance.

5 Proofs of the main results

5.1 Preliminary results

Without loss of generality, throughout this section we assume $J^* = 0$. An important step of the proof is the representation of solutions of system (11) with controls (13) and initial

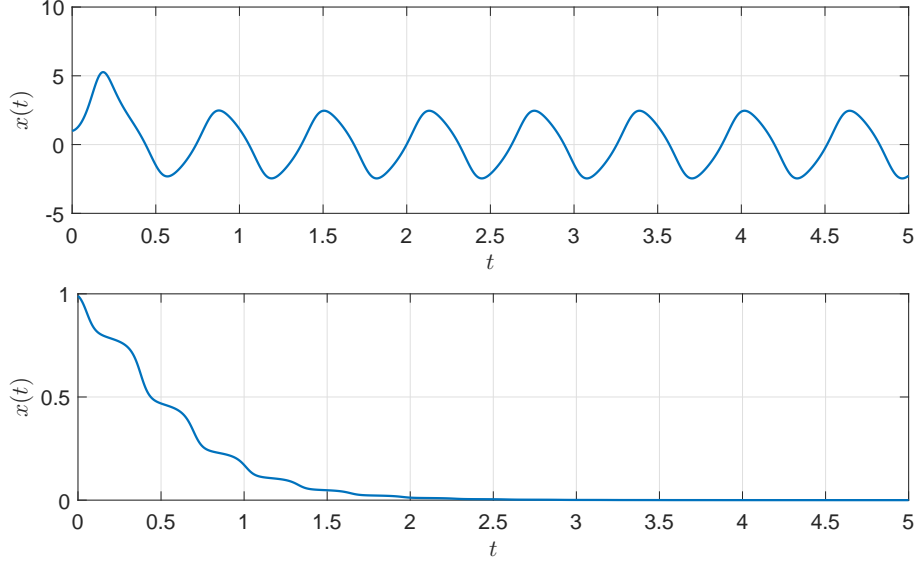


Figure 5: Trajectory of (23) with the control law (20) (top) and (24) (bottom); $\alpha = 4$, $\omega = 10$, $\mu = 1$.

conditions $x(0) = x^0 \in D$ by using the Volterra series [15, 22]. By taking into account formula (7), it can be written as follows:

$$\begin{aligned} x(\varepsilon) &= x^0 + \frac{1}{2} \sum_{i=1}^n [F_{1i}e_i, F_{2i}e_i](J(x^0)) \int_0^\varepsilon \int_0^\tau (u_{2i}(\tau)u_{1i}(\theta) - u_{1i}(\tau)u_{2i}(\theta)) d\theta d\tau + R(\varepsilon) \\ &= x^0 - \varepsilon \sum_{i=1}^n \frac{\partial J(x^0)}{\partial x_i} F_{0i}(J(x^0))e_i + R(\varepsilon), \end{aligned} \quad (25)$$

where $R(\varepsilon)$ is the remainder of the Volterra series expansion. Denote

$$\nu = \max_t \sum_{\substack{i=1 \\ s=1,2}}^n |u_{si}^\varepsilon(t)| = 2\sqrt{2\pi}\varepsilon^{-1/2} \sum_{i=1}^n \sqrt{k_i}. \quad (26)$$

To prove Theorems 3 and 4, we will use the following three lemmas (the proof of each lemma can be found in the Appendix).

Lemma 2. *Let $D \subset \mathbb{R}^n$ be a bounded convex domain, $V, h_i : D \rightarrow \mathbb{R}$, $i = \overline{1, n}$, $V \in C^2(D; \mathbb{R})$, $x^* \in D$, and the following inequalities hold:*

$$\begin{aligned} \gamma_1 \|x - x^*\|^{2m_1} &\leq V(x) \leq \gamma_2 \|x - x^*\|^{2m_1}, \\ \kappa_1 V(x)^{2-\frac{1}{m_1}} &\leq \|\nabla V(x)\|^2 \leq \kappa_2 V(x)^{2-\frac{1}{m_1}}, \\ \left\| \frac{\partial^2 V(x)}{\partial x^2} \right\| &\leq \mu V(x)^{1-\frac{1}{m_1}}, \\ \alpha_1 V(x)^{m_2} &\leq h_i(x) \leq \alpha_2 V(x)^{m_2}, \text{ for all } x \in D, \end{aligned}$$

where $m_1 \geq 1$, $m_2 \geq 0$, and $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \kappa_1, \kappa_2, \mu$ are positive constants. Then, for any $x^0 \in D \setminus \{x^*\}$ and a function $x : [0, \varepsilon] \rightarrow D$ satisfying the conditions

$$x(0)=x^0, \quad x(\varepsilon)=x^0-\varepsilon \sum_{i=1}^n \frac{\partial V(x^0)}{\partial x_i} h_i(x^0)e_i+r_\varepsilon, \quad r_\varepsilon \in \mathbb{R}^n,$$

the following properties hold:

- if $m_1 = 1$ and $m_2 = 0$, then

$$V(x(\varepsilon)) \leq V(x^0) \left(1 - \left(\alpha_1 \kappa_1 \varepsilon - \frac{\sqrt{\kappa_2} \|r_\varepsilon\|}{\sqrt{V(x^0)}} - \frac{\mu}{2} \left(\alpha_2 \sqrt{\kappa_2} \varepsilon + \frac{\|r_\varepsilon\|}{\sqrt{V(x^0)}} \right)^2 \right) \right);$$

- if $m_1 > 1$ or $m_2 > 0$, and $x^* \notin \text{co}\{x^0, x(\varepsilon)\}$, then

$$V^{\tilde{m}}(x(\varepsilon)) \leq V^{\tilde{m}}(x^0) \left(1 - \tilde{m} V^{\tilde{m}}(x^0) \left(\alpha_1 \kappa_1 \varepsilon - \frac{\sqrt{\kappa_2} \|r_\varepsilon\|}{V^{\tilde{m} + \frac{1}{2m_1}}(x^0)} - \frac{\bar{\mu}}{2} \left(\alpha_2 \sqrt{\kappa_2} \varepsilon + \frac{\|r_\varepsilon\|}{V^{\tilde{m} + \frac{1}{2m_1}}(x^0)} \right)^2 \right) \right),$$

where $\bar{\mu} = (\mu + \kappa_2(\tilde{m} - 1))\gamma_2^{\tilde{m} - \frac{1}{m_1}} V^{\frac{1}{m_1}}(x^0) \sup_{\eta \in D} \|\eta\|^{2(m_1(1+m_2)-2)}$,

$\xi = x^0 - \theta \left(\varepsilon \sum_{i=1}^n \frac{\partial V(x^0)}{\partial x_i} h_i(x^0) e_i - r_\varepsilon \right)$ for some $\theta \in (0, 1)$.

Lemma 3. Let $D \subset \mathbb{R}^n$ be a convex domain, $x(t) \in D$, $0 \leq t \leq \tau$, be a solution of system (11) with controls (13). Assume that there exist positive constants M, L such that

$$\begin{aligned} |F_{si}(J(x))| &\leq M \|x - x^*\|^m, \\ |F_{si}(J(x)) - F_{si}(J(y))| &\leq L \|x - y\| \end{aligned}$$

with some $m \geq 0$, for all $x, y \in D$, $s=1, 2$, $i=\overline{1, n}$. Then

$$\|x(t) - x(0)\| \leq \frac{M}{L} \|x(0) - x^*\|^m (e^{\nu L t} - 1), \quad t \in [0, \tau], \quad (27)$$

with ν defined in (26).

Lemma 4. Let $D \subset \mathbb{R}^n$ be a convex domain, and let $x(t) \in D$, $0 \leq t \leq \tau$, be a solution of system (11) with the initial value $x(0) = x^0 \in D$ and controls (13). Suppose that $F_{si}(J(\cdot)) \in C^2(D; \mathbb{R})$ satisfy

$$\|\nabla F_{si}(J(x))\| \leq L, \quad \left\| \frac{\partial^2 F_{si}(J(x))}{\partial x^2} \right\| \leq H, \quad s=1, 2, i=\overline{1, n},$$

for all $x \in D$, with some constants $H, L > 0$, and, moreover, there exist $M, \tilde{L} > 0$, $m \geq 0$ such that

$$|F_{si}(J(x))| \leq M \|x - x^*\|^m \text{ for all } x \in D,$$

and, in case $m \geq 1$,

$$\|\nabla F_{si}(J(x))\| \leq \tilde{L} \|x - x^*\|^{m-1}.$$

Then the remainder $R(\tau)$ of the Volterra expansion (25) of $x(t)$ satisfies the following estimate:

$$\begin{aligned} \|R(\tau)\| &\leq \frac{M\tilde{L}\|x^0 - x^*\|^{2m-1}}{L^2} \left(-\frac{1}{2} \left((L\nu\tau + 1)^2 + 1 \right) + e^{L\nu\tau} \right) \\ &+ \frac{HM^2\sqrt{n}\|x^0 - x^*\|^{2m}}{4L^3} \left(\left(e^{L\nu\tau} - 2 \right)^2 + 2L\nu\tau - 1 \right) \\ &= \|x^0 - x^*\|^{2m-1} \left(\frac{M(L\tilde{L} + HM\|x^0 - x^*\|\sqrt{n})}{6} \nu^3 \tau^3 + O(\nu^4 \tau^4) \right), \end{aligned}$$

with ν defined in (26).

5.2 Proof of Theorem 3

For $\Delta > 0$ defined from Assumption 1 and a given $\rho \in (0, \Delta)$, fix $\delta_0 \in (0, \Delta)$ and $0 < \rho_1 < \rho_2 < \rho$. Without loss of generality, we assume $\rho < \min\{\delta_0, \hat{\rho}\}$, where $\hat{\rho}$ will be specified later. Denote

$$D_0 = \overline{B_{\delta_0}(x^*) \setminus B_{\rho_2}(x^*)} \subset D = B_{\Delta}(x^*),$$

$$M = \sup_{\substack{x \in \overline{B_{\delta_0}(x^*)} \\ s=1,2,1 \leq i \leq n}} |F_{si}(J(x))|,$$

Step 1. Let us specify a positive number ε_0 such that all solutions $x(t)$ of system (11) with controls (13) and initial conditions $x(0) = x^0 \in D_0$ are well defined on $t \in [0, \varepsilon]$ for each $\varepsilon \in (0, \varepsilon_0]$. With this purpose, put

$$d = \min\{\Delta - \delta_0, \rho - \rho_2, \rho_2 - \rho_1\}.$$

Then $\|x(t) - x^0\| < d$, $t \in [0, \varepsilon]$, by Lemma 3 with $m = 0$, for each $\varepsilon \in (0, \varepsilon_0]$, where

$$0 < \varepsilon_0 < \left(2\sqrt{2\pi}L \sum_{i=1}^n \sqrt{k_i}\right)^{-2} \ln^2 \left(\frac{Ld}{M} + 1\right). \quad (28)$$

Hence, all solutions $x(t)$ of system (11) with the initial conditions $x^0 \in D_0$ and controls $u_{si}^\varepsilon(t)$ (13) are in the set D for $t \in [0, \varepsilon]$. Furthermore, with such choice of d , the following properties hold:

- P1 $x^0 \in \overline{B_{\rho_2}(x^*)} \Rightarrow x(t) \in B_{\rho}(x^*)$ for all $t \in [0, \varepsilon]$;
- P2 $x^0 \in D_0 \Rightarrow x(t) \in D \setminus \overline{B_{\rho_1}(x^*)}$ for all $t \in [0, \varepsilon]$.

Step 2. Let J satisfy the conditions of Theorem 3. We introduce the level sets

$$\mathcal{L}_c = \{x \in D : J(x) \leq c\},$$

and define

$$c_0 = \inf_{x \in D \setminus \overline{B_{\delta_0}(x^*)}} J(x) > 0, \quad \delta = \inf_{x \in D \setminus \mathcal{L}_{c_0}} \|x - x^*\| > 0.$$

Coming back to the choice of $\hat{\rho}$ in Step 1, assume $\hat{\rho} \in (0, \delta)$. It is easy to see that $\delta \leq \delta_0$, $\mathcal{L}_c \subseteq \mathcal{L}_{c_0}$ for all $c \leq c_0$, and, since $\rho < \hat{\rho}$,

$$\overline{B_{\rho}(x^*)} \subset \overline{B_{\hat{\rho}}(x^*)} \subset \overline{B_{\delta}(x^*)} \subseteq \mathcal{L}_{c_0} \subseteq \overline{B_{\delta_0}(x^*)}.$$

Step 3. Case $m_1 = 1$. The proof of this case goes along the line of proof [22, Theorem 2.2] with slight modifications. Note that the results of [22] cannot be directly applied since, first, F_{si} may have unbounded second derivatives, and second, F_{si} do not necessary tend to 0 as $x \rightarrow x^*$.

We will show that, for $\varepsilon > 0$ small enough, there exists a positive $\lambda = \lambda(\varepsilon) \in (0, \varepsilon^{-1})$ such that

$$J(x(\varepsilon)) \leq (1 - \varepsilon\lambda)J(x^0), \quad (29)$$

for any solution of system (11) with the initial condition $x^0 \in \mathcal{L}_{c_0} \setminus B_{\rho_2}\{x^*\} \subseteq D_0$ and controls $u_{si}^\varepsilon(t)$ given by (13). Recall that such solutions can be represented in the form (25), and define $L, H > 0$ such that

$$\|\nabla F_{si}(J(x))\| \leq L \text{ for all } x \in \overline{B_{\Delta}(x^*)},$$

$$\left\| \frac{\partial^2 F_{si}(J(x))}{\partial x^2} \right\| \leq H \text{ for all } x \in D \setminus B_{\rho_1}(x^*),$$

for $s = 1, 2$, $i = \overline{1, n}$. Then taking into account P2 and applying Lemma 4 with $V = J$, $m = 0$, we estimate the remainder in formula (25) as

$$\|R(\varepsilon)\| \leq \Omega \varepsilon^{3/2}, \quad (30)$$

with some positive constant Ω , provided that $\nu\varepsilon < 1$, or equivalently,

$$\varepsilon \leq \varepsilon_1 = \left(2\sqrt{2\pi} \sum_{i=1}^n \sqrt{k_i}\right)^{-2}.$$

Thus, all the conditions of Lemma 2 are satisfied with $V=J$, $m_1=1$, $m_2=0$, and we apply it for estimating the decay rate of the cost function along the solutions of system (11):

$$J(x(\varepsilon)) \leq J(x^0) \left(1 - \left(\alpha_1 \kappa_1 \varepsilon - \frac{\sqrt{\kappa_2} \Omega \varepsilon^{3/2}}{\sqrt{J(x^0)}} - \frac{\mu}{2} \left(\alpha_2 \sqrt{\kappa_2} \varepsilon + \frac{\Omega \varepsilon^{3/2}}{\sqrt{J(x^0)}}\right)^2\right)\right).$$

Taking into account $x^0 \in D_0$ and assumption A3, we have

$$J(x^0) \geq \gamma_1 \rho_2^{2m_1}. \quad (31)$$

Thus, to obtain (29) it suffices to define $\varepsilon_{\max}, \lambda_{\max}$ such that, for all $x^0 \in D_0$, $\varepsilon \in (0, \varepsilon_{\max}]$, $\lambda \in (0, \lambda_{\max}]$,

$$\alpha_1 \kappa_1 - \frac{\sqrt{\kappa_2} \Omega \varepsilon^{1/2}}{\sqrt{\gamma_1} \rho_2^{m_1}} - \frac{\mu \varepsilon}{2} \left(\sqrt{\kappa_2} + \frac{\Omega \varepsilon^{1/2}}{\sqrt{\gamma_1} \rho_2^{m_1}}\right)^2 \geq \lambda. \quad (32)$$

Since $\alpha_1 \kappa_1 > 0$, we conclude that there exist $\varepsilon_{\max} > 0$ and $\lambda_{\max} \in (0, \alpha_1 \kappa_1)$ such that inequality (32) holds for all $\varepsilon \in (0, \varepsilon_{\max}]$, $\lambda \in (0, \lambda_{\max}]$. Defining $\bar{\varepsilon} = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_{\max}\}$, we conclude that inequality (29) is satisfied for any $\varepsilon \in (0, \bar{\varepsilon}]$ with $\lambda = \lambda(\varepsilon) = \min\{\lambda_{\max}, \varepsilon^{-1}\}$ provided that $x^0 \in \mathcal{L}_{c_0} \setminus B_{\rho_2}(x^*) \subseteq D_0$. Besides, $\lambda \varepsilon < 1$, and estimate (29) can be rewritten as

$$J(x(\varepsilon)) \leq J(x^0) e^{-\lambda \varepsilon} \text{ for } x^0 \in \mathcal{L}_{c_0} \setminus B_{\rho_2}(x^*). \quad (33)$$

i) Hence, we conclude that if $x^0 \in B_\delta(x^*) \setminus B_{\rho_2}(x^*)$, $\varepsilon \in (0, \bar{\varepsilon}]$, and $u_{si}^\varepsilon(t)$ are given by (13), then the corresponding solution of system (11) satisfy one of the two cases:

- a) $x(n\varepsilon) \in \mathcal{L}_{c_0} \setminus B_{\rho_2}(x^*) \subseteq D_0$ for $n = 0, 1, \dots$,
- b) $x(n\varepsilon) \in \mathcal{L}_{c_0} \setminus B_{\rho_2}(x^*) \subseteq D_0$ for $n = 0, 1, \dots, N$,
and $x((N+1)\varepsilon) \in B_{\rho_2}(x^*)$ with some $N \in \mathbb{N}$.

Consider case b) (the study of case a) is similar). From P2 we conclude that $x((N+1)\varepsilon) \in B_{\rho_2}(x^*) \setminus B_{\rho_1}(x^*)$. Iterating inequality (33) for $x^0 \in \mathcal{L}_{c_0} \setminus B_{\rho_2}(x^*)$, we get

$$J(x(t)) \leq J(x^0) e^{-\lambda t} \text{ for } t = 0, \varepsilon, \dots, N\varepsilon.$$

Then from A3,

$$\|x(t) - x^*\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}} \|x^0 - x^*\| e^{-\lambda t} \text{ if } t = 0, \varepsilon, \dots, N\varepsilon. \quad (34)$$

For any $t \geq 0$, denote the integer part of $\frac{t}{\varepsilon}$ as t_{in}^ε . Then

$$\begin{aligned} \|x(t) - x^*\| &= \|x(t) - x^* - x(t_{in}^\varepsilon \varepsilon) + x(t_{in}^\varepsilon \varepsilon)\| \\ &\leq \|x(t_{in}^\varepsilon \varepsilon) - x^*\| + \|x(t) - x(t_{in}^\varepsilon \varepsilon)\|, \end{aligned} \quad (35)$$

for $t \in [0, (N+1)\varepsilon]$. Since $t - t_{in}^\varepsilon < \varepsilon$, we may apply Lemma 3 to estimate the second term in the last inequality; besides, the first term can be estimated from (34). Hence,

$$\|x(t) - x^*\| \leq \sqrt{\frac{\gamma_2}{\gamma_1}} \|x^0 - x^*\| e^{-\lambda t_{in}^\varepsilon} + \frac{M}{L} (e^{\nu L \varepsilon} - 1), \quad (36)$$

for $t \in [0, (N+1)\varepsilon]$. Since N is defined in such a way that $x((N+1)\varepsilon) \in B_{\rho_2}(x^*)$, we conclude that

$$\begin{aligned} \|x(t) - x^*\| &\leq \sigma \|x^0 - x^*\| e^{-\lambda t} + \psi_\varepsilon, \text{ for } t \in [0, (N+1)\varepsilon], \\ \|x(t) - x^*\| &< \rho_2 \text{ for } t = (N+1)\varepsilon, \end{aligned}$$

where $\sigma = \sqrt{\frac{\gamma_2}{\gamma_1}} e^{\lambda \varepsilon}$, $\psi_\varepsilon = \frac{M}{L} (e^{\nu L \varepsilon} - 1) < \rho$ due to the choice of ε_0 in (28) and d .

ii) It remains to show that

$$x^0 \in B_{\rho_2}(x^*) \Rightarrow x^0 \in B_\rho(x^*) \text{ for all } t \geq 0.$$

Iterating P1, we have again two cases: either $x(n\varepsilon) \in B_{\rho_2}(x^*)$ for all $n = 0, 1, \dots$ or there exists an $\bar{N} \in \mathbb{N}$ such that $x(\bar{N}\varepsilon) \in B_\rho(x^*) \setminus B_{\rho_2}(x^*)$. It is easy to see that in the first case $x(t) \in B_\rho(x^*)$ for all $t \geq 0$. In the second case, we observe that $B_\rho(x^*) \subseteq \mathcal{L}_{c_0}$ and apply the above argumentation (part i)).

Case $m_1 > 1$. The general line of the proof is similar to the case $m_1 = 1$, and the main modifications concern the decay rate of the solutions which can be estimated from the second assertion of Lemma 2. In particular, instead of (33), the following inequality can be obtained for small enough ε :

$$J(x(\varepsilon)) \leq (J^{-\tilde{m}}(x^0) + \lambda \varepsilon)^{-\frac{1}{\tilde{m}}}, \quad (37)$$

with $\tilde{m} = \frac{m_1 - 1}{m_1} > 0$ and some positive constant $\lambda > 0$. This fact will be proven below. Then we use the same argumentation as in the case $m_1 = 1$ with modified estimate (34) which follows by iterating inequality (37) for $x^0 \in \mathcal{L}_{c_0} \setminus B_{\rho_2}(x^*)$ and applying A3:

$$\begin{aligned} \|x(t) - x^*\| &\leq (\sigma_1 \|x^0 - x^*\|^{2(1-m_1)} + \sigma_2 t)^{\frac{1}{2(1-m_1)}} \\ &\text{for } t = 0, \varepsilon, \dots, N\varepsilon, \end{aligned} \quad (38)$$

where $\sigma_1 = (\gamma_1/\gamma_2)^{\tilde{m}}$, $\sigma_2 = \lambda \gamma_1^{\tilde{m}} > 0$. Consequently, estimating each term in (35) with Lemma 3 and (38), we get

$$\begin{aligned} \|x(t) - x^*\| &= (\sigma_1 \|x^0 - x^*\|^{2(1-m_1)} + \sigma_2 t_{in} \varepsilon)^{\frac{1}{2(1-m_1)}} \\ &+ \frac{M}{L} (e^{\nu L \varepsilon} - 1) \leq \varphi(\|x^0 - x^*\|, t, \varepsilon) + \frac{M}{L} (e^{\nu L \varepsilon} - 1), \end{aligned} \quad (39)$$

for $t \in [0, (N+1)\varepsilon]$, where $\varphi = (\sigma_1 \|x^0 - x^*\|^{2(1-m_1)} - \sigma_2 \varepsilon + \sigma_2 t)^{\frac{1}{2(1-m_1)}}$. Then the argumentation similar to the one after (36) completes the proof.

Thus, it remains to show that there exist $\varepsilon_{\max}, \lambda > 0$ such that inequality (37) holds for all $x^0 \in D_0$, $\varepsilon \in (0, \varepsilon_{\max}]$. By the definition of ε_0 , if $x^0 \in D_0$ then P2 holds and $x(t) \in B_{\rho_2 - \rho_1}(x^0)$ for $t \in [0, \varepsilon]$. Thus, $x^* \notin \text{co}\{x^0, x(\varepsilon)\}$. Applying formula (25), estimates (30), (31), and Lemma 2 with $V = J$, $m_1 > 1$, $m_2 = 0$, we obtain

$$\begin{aligned} J^{\tilde{m}}(x(\varepsilon)) &\leq J^{2\tilde{m}}(x^0) \left(J^{-\tilde{m}}(x^0) - \varepsilon \tilde{m} \left(\alpha_1 \kappa_1 - \frac{\sqrt{\varepsilon \kappa_2} \Omega}{\gamma_1^{1 - \frac{1}{2m_1}} \rho_2^{2m_1 - 1}} - \frac{\bar{\mu} \varepsilon}{2} \left(\alpha_2 \sqrt{\kappa_2} \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{\sqrt{\varepsilon} \Omega}{\gamma_1^{1 - \frac{1}{2m_1}} \rho_2^{2m_1 - 1}} \right)^2 \right) \right), \end{aligned}$$

where $\bar{\mu} = (\mu + \kappa_2(\tilde{m} - 1))\gamma_2^{\tilde{m} - \frac{1}{m_1}} J^{\frac{1}{m_1}}(x^0) \Delta^{2(m_1-2)}$, $\tilde{m} = 1 - \frac{1}{m_1}$, $\xi = x^0 - \theta(\varepsilon \sum_{i=1}^n \frac{\partial J(x^0)}{\partial x_i} F_{i0}(x^0) e_i - r_\varepsilon)$ for some $\theta \in (0, 1)$. Hence,

$$J^{\tilde{m}}(x(\varepsilon)) \leq J^{2\tilde{m}}(x^0) \left(J^{-\tilde{m}}(x^0) - \varepsilon \bar{\lambda} \right) \quad (40)$$

$$\text{with } \bar{\lambda} = \tilde{m} \left(\alpha_1 \kappa_1 - \frac{\sqrt{\varepsilon \kappa_2} \Omega}{\gamma_1^{1 - \frac{1}{2m_1}} \rho_2^{2m_1-1}} - \frac{\varepsilon \bar{\mu}}{2} \left(\alpha_2 \sqrt{\kappa_2} + \frac{\sqrt{\varepsilon} \Omega}{\gamma_1^{1 - \frac{1}{2m_1}} \rho_2^{2m_1-1}} \right)^2 \right).$$

Similarly to the case $m_1=1$, there exist $\varepsilon_{\max}, \lambda_{\max}$ such that inequality (40) holds for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\bar{\lambda} \in (0, \lambda_{\max}]$.

Then we have with $\bar{\varepsilon} = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_{\max}\}$, $\lambda = \lambda(\varepsilon) = \min\{\tilde{m} \lambda_{\max}, (\varepsilon J^{\tilde{m}}(x^0))^{-1}\}$,

$$\begin{aligned} J^{\tilde{m}}(x(\varepsilon)) &\leq J^{2\tilde{m}}(x^0) \left(J^{-\tilde{m}}(x^0) - \varepsilon \lambda \right) \\ &= \frac{1 - \varepsilon^2 \lambda^2 J^{2\tilde{m}}(x^0)}{J^{-\tilde{m}}(x^0) + \varepsilon \lambda} \leq \left(J^{-\tilde{m}}(x^0) + \varepsilon \lambda \right)^{-1}. \end{aligned} \quad (41)$$

The last inequality yields (37) provided that $x^0 \in \mathcal{L}_{c_0} \setminus B_{\rho_2}$. Note that it follows from (37) that $J(x(\varepsilon)) \leq J(x^0)$, therefore, the coefficients $\lambda, \bar{\mu}$ can be chosen independently of x^0 , δ by estimating $J(x^0) \leq \gamma_2 \Delta^{2m_1}$.

5.3 Proof of Theorem 4

Step 1. Let $\Delta > 0$ be defined as in Assumptions 1,2. For a fixed $\delta_0 \in (0, \Delta)$, denote $D_0 = \overline{B_{\delta_0}(x^*)} \subset D = B_\Delta(x^*)$, and chose a $\varepsilon_0 > 0$ such that all solutions of system (11) are well defined on $t \in [0, \varepsilon]$ for each $\varepsilon \in (0, \varepsilon_0]$, provided that $x^0 = x(0) \in D_0$ and u_{si}^ε are given by (13), $s=1, 2, i=\overline{1, n}$. For this purpose, define $d = \Delta - \delta_0 > 0$. If $\Delta = +\infty$, then we take $d = +\infty$ and arbitrary $\varepsilon_0 \in (0, \infty)$, otherwise we choose $\varepsilon_0 \in \left(0, \left(\frac{1}{2\sqrt{2\pi} L \sum_{i=1}^n \sqrt{k_i}} \ln \left(\frac{Ld}{M \Delta^{m_1(1+m_2)} + 1} \right) + 1 \right)^2 \right)$, where M, L, m_1, m_2 are defined as in A4 and Assumption 2. Thus, by Lemma 3, all solutions $x(t)$ of system (11) with the initial conditions $x^0 \in D_0$ and controls u_{si}^ε given by (13) are in the set D for $t \in [0, \varepsilon]$.

Step 2. Let J satisfy the conditions of the theorem. We introduce the level sets $\mathcal{L}_c = \{x \in D : J(x) \leq c\}$, and define $c_0 = \inf_{x \in D \setminus \overline{B_{\delta_0}(x^*)}} J(x) > 0$, $\delta = \inf_{x \in D \setminus \mathcal{L}_{c_0}} \|x - x^*\| > 0$. It is easy to

see that $\delta \leq \delta_0$, $\overline{B_\delta(x^*)} \subseteq \mathcal{L}_{c_0} \subseteq D_0$, and $\mathcal{L}_c \subseteq \mathcal{L}_{c_0}$ for all $c \leq c_0$.

Step 3. To estimate the remainder in (25), we use Assumption 2 and Lemma 4 with $m = m_1(1+m_2)$: there is an $\tilde{\Omega} > 0$ such that $\|R(\varepsilon)\| \leq \tilde{\Omega} \|x^0 - x^*\|^{2m_1(1+m_2)-1}$ provided that $\varepsilon \leq \varepsilon_1 = (2\sqrt{2\pi} \sum_{i=1}^n \sqrt{k_i})^{-2}$. Then from A3,

$$\|R(\varepsilon)\| \leq \Omega J^{\tilde{m} + \frac{1}{2m_1}}(x^0) \varepsilon^{3/2},$$

with $\tilde{m} = 1 + m_2 - \frac{1}{m_1}$, $\Omega = \tilde{\Omega} \gamma_1^{-\tilde{m} - \frac{1}{2m_1}}$.

To apply Lemma 2, we first show that $x^* \notin \text{co}\{x^0, x(\varepsilon)\}$ for ε small enough. Define $\varepsilon_2 > 0$ such that

$$\varepsilon_2 (\alpha_2 \sqrt{\kappa_2} + \sqrt{\varepsilon_2} \Omega) < \gamma_2^{-\frac{1}{2m_1}} \delta_0^{-\tilde{m}}. \quad (42)$$

Note that the above ε_2 may be chosen independently of $x^0 \in D_0$ as $D_0 \subset D$ is a bounded set and J satisfies A3. Then we show that, for all $x^0 \in D_0$, $\varepsilon \in (0, \varepsilon_2]$, $\theta \in (0, 1)$, the

vector $\eta = (1 - \theta)x^0 + \theta x(\varepsilon)$ satisfies $\|\eta - x^*\| > 0$. Indeed, the triangle inequality implies that $\|\eta - x^*\| \geq \|x^0 - x^*\| - \theta\|x^0 - x(\varepsilon)\|$, and the representation (25) with $r_\varepsilon = R(\varepsilon)$ yields

$$\begin{aligned} \|\eta - x^*\| &\geq \|x^0 - x^*\| - \theta \left\| \varepsilon \sum_{i=1}^n \frac{\partial J(x^0)}{\partial x_i} F_{0i}(x^0) e_i - r_\varepsilon \right\| \\ &\geq \gamma_2^{-\frac{1}{2m_1}} J^{\frac{1}{2m_1}}(x^0) - \varepsilon J^{\tilde{m} + \frac{1}{2m_1}}(x^0) (\alpha_2 \sqrt{\kappa_2} + \sqrt{\varepsilon} \Omega) \\ &\geq J^{\tilde{m} + \frac{1}{2m_1}}(x^0) \left(\gamma_2^{-\frac{1}{2m_1}} \delta_0^{-\tilde{m}} - \varepsilon (\alpha_2 \sqrt{\kappa_2} + \sqrt{\varepsilon} \Omega) \right) > 0, \end{aligned}$$

where we have used condition (42). Thus, $x^* \notin \text{co}\{x^0, x(\varepsilon)\}$ and Lemma 2 with $V = J$, $m_1 > 1$, $m_2 > 0$ yields

$$J^{\tilde{m}}(x(\varepsilon)) \leq J^{2\tilde{m}}(x^0) \left(J^{-\tilde{m}}(x^0) - \varepsilon \tilde{m} \left(\alpha_1 \kappa_1 - \sqrt{\varepsilon \kappa_2} \Omega - \frac{\bar{\mu} \varepsilon}{2} \left(\alpha_2 \sqrt{\kappa_2} + \sqrt{\varepsilon} \Omega \right)^2 \right) \right),$$

with $\bar{\mu} = (\mu + \kappa_2(\tilde{m} - 1)) \gamma_2^{\tilde{m} - \frac{1}{m_1}} J^{\frac{1}{m_1}}(x^0) \Delta^{2(m_1(1+m_2)-2)}$, $\xi = x^0 - \theta \left(\varepsilon \sum_{i=1}^n \frac{\partial J(x^0)}{\partial x_i} F_{i0}(x^0) e_i - r_\varepsilon \right)$ for some $\theta \in (0, 1)$. Hence,

$$J^{\tilde{m}}(x(\varepsilon)) \leq J^{2\tilde{m}}(x^0) \left(J^{-\tilde{m}}(x^0) - \varepsilon \bar{\lambda} \tilde{m} \right),$$

where $\bar{\lambda} = \tilde{m} \left(\alpha_1 \kappa_1 - \sqrt{\varepsilon \kappa_2} \Omega - \frac{\bar{\mu} \varepsilon}{2} \left(\alpha_2 \sqrt{\kappa_2} + \sqrt{\varepsilon} \Omega \right)^2 \right)$.

Since $\alpha_1 \kappa_1 > 0$, we conclude that there exist $\varepsilon_{\max} > 0$ and $\lambda_{\max} > 0$ such that the inequality $\bar{\lambda} > \lambda$ holds for all $\varepsilon \in (0, \varepsilon_{\max}]$, $\lambda \in (0, \lambda_{\max}]$. Then we choose $\bar{\varepsilon} = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_{\max}\}$ and conclude that, for each $\varepsilon \in (0, \bar{\varepsilon}]$ and for $\lambda = \lambda(\varepsilon) = \min\{\lambda_{\max}, (\varepsilon J^{\tilde{m}}(x^0))^{-1}\}$, the solutions of (11) with u_{si}^ε given by (13) and the initial conditions $x^0 \in \mathcal{L}_{c_0}$ possess the property $J^{\tilde{m}}(x(\varepsilon)) \leq J^{2\tilde{m}}(x^0) \left(J^{-\tilde{m}}(x^0) - \varepsilon \lambda \right)$, or, similarly to (41),

$$J(x(\varepsilon)) \leq (J^{-\tilde{m}}(x^0) + \varepsilon \lambda)^{-\frac{1}{\tilde{m}}}. \quad (43)$$

Note that, since $J(x(\varepsilon)) \leq J(x^0)$, the same $\lambda, \bar{\mu}$ can be chosen for all $x^0 \in D_0$ by estimating $J(x^0) \leq \gamma_1 \Delta^{2m_1}$. Thus, if $x^0 \in B_\delta(x^*) \subseteq \mathcal{L}_{c_0}$, $\varepsilon \in (0, \bar{\varepsilon}]$, and u_{si}^ε are given by (13), then the corresponding solution of (11) is well defined in D : $x(n\varepsilon) \in \mathcal{L}_{c_0} \subseteq D_0$ for $n = 0, 1, 2, \dots$, and due to the choice of d, ε and Lemma 3 $x(t) \in D$ for all $t \geq 0$. By iterating (43) for $x^0 \in B_\delta(x^*) \subseteq \mathcal{L}_{c_0}$ and taking into account A3, we conclude that

$$\|x(t) - x^*\| \leq (\sigma_1 \|x^0 - x^*\|^{-2\tilde{m}m_1} + \sigma_2 t)^{\frac{1}{-2\tilde{m}m_1}} \quad (44)$$

for $t = 0, \varepsilon, 2\varepsilon, \dots$, with positive constants $\sigma_1 = \left(\frac{\gamma_1}{\gamma_2} \right)^{\tilde{m}}$, $\sigma_2 = \lambda \gamma_1^{\tilde{m}}$. Similarly to the proof of Theorem 3, for any $t \geq 0$ denote the integer part of $\frac{t}{\varepsilon}$ as t_{in}^ε , and note that $t - t_{in}^\varepsilon < \varepsilon$. By using the triangle inequality, estimate (44) and Lemma 3 with $m = m_1(1 + m_2)$, we obtain

$$\begin{aligned} \|x(t) - x^*\| &= \|x(t) - x^* - x(t_{in}^\varepsilon \varepsilon) + x(t_{in}^\varepsilon \varepsilon)\| \\ &\leq \|x(t_{in}^\varepsilon \varepsilon) - x^*\| + \|x(t) - x(t_{in}^\varepsilon \varepsilon)\| \\ &\leq \|x(t_{in}^\varepsilon \varepsilon) - x^*\| + \frac{M}{L} \|x(t_{in}^\varepsilon \varepsilon) - x^*\|^{m_1(1+m_2)} (e^{\nu L \varepsilon} - 1) \\ &= O\left(t^{-\frac{1}{2m_1\tilde{m}}}\right) + O\left(t^{-\frac{m_1(1+m_2)}{2m_1\tilde{m}}}\right) \text{ as } t \rightarrow +\infty. \end{aligned}$$

The last estimate yields

$$\|x(t) - x^*\| = O\left(t^{-\frac{1}{2(m_1(1+m_2)-1)}}\right) \text{ as } t \rightarrow +\infty.$$

6 Conclusions

In this paper, we have proposed a new formula for constructing a class of vector fields to approximate gradient-like flows based on the Lie bracket approximation idea. We have shown how this formula gives rise to a broad class of controls for the extremum seeking and vibrational stabilization problems. It generalizes and unifies some existing results and gives an opportunity for the synthesis of new control functions. While the formula looks rather simple, we believe that it potentially comprises more applications than the ones discussed in this paper. Furthermore, from a conceptual point of view, we have presented a novel approach to the proof of asymptotic stability properties of extremum seeking systems. This approach gives several advantages compared to the existing results. First, the proofs of the main results present a constructive procedure for defining the frequencies of the control functions for ensuring the practical asymptotic stability; second, the practical *exponential* stability is proven for certain cost functions. Finally, the main advantage of the developed approach are conditions for the asymptotic stability *in the sense of Lyapunov* for extremum seeking systems whose vector fields satisfy certain additional requirements. The important step in the proof of this result are novel decay rate estimates for the cost function along the solutions of the obtained extremum seeking system. Besides, some auxiliary results of this paper (in particular, Lemmas 2–4) extend the results of [22, 23] and can be exploited in other control problems, e.g., asymptotic stabilization of nonholonomic systems when the exponential stabilization is not possible.

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A Proofs of auxiliary results

A.1 Proof of Lemma 2

For $m_1 = 1$ and $m_2 = 0$ the lemma has been proven in [22]. Consider the case $\tilde{m} = \frac{m_1(1+m_2)-1}{m_1} > 0$.

Denote $x^0 = x(0) \in D \setminus \{x^*\}$, $W(x) = V^{\tilde{m}}(x)$,

$$y = x(\varepsilon) - x^0 = -\varepsilon \sum_{i=1}^n \frac{\partial V(x^0)}{\partial x_i} h_i(x^0) e_i + r_\varepsilon,$$

and observe that $\nabla W(x) = \tilde{m} V(x)^{\tilde{m}-1} \nabla V(x)$,

$$\left\| \frac{\partial^2 W(x)}{\partial x^2} \right\| \leq \tilde{m} ((\tilde{m} - 1)\kappa_2 + \mu) V^{\tilde{m}-\frac{1}{m_1}}(x), \quad x \neq x^*.$$

Applying Taylor's theorem for the function $W(x^0 + y)$ with the Lagrange form of the remainder, we obtain

$$W(x(\varepsilon)) = W(x^0) + \tilde{m} V^{\tilde{m}-1}(x^0) \sum_{i=1}^n \frac{\partial V(x^0)}{\partial x_i} y_i + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 W(\xi)}{\partial x_i \partial x_j} y_i y_j,$$

where $\xi = x^0 - \theta(\varepsilon \sum_{i=1}^n \frac{\partial V(x^0)}{\partial x_i} h_i(x^0) e_i - r_\varepsilon)$ for some $\theta \in (0, 1)$. Note that, from the assumptions of the lemma, $\xi = (1 - \theta)x^0 + \theta x(\varepsilon) \in D \setminus \{x^*\}$. Then using the Cauchy-Schwarz inequality and exploiting the conditions of Lemma 2, we get the following estimate:

$$\begin{aligned} W(x(\varepsilon)) &\leq W(x^0) + \tilde{m} V^{2\tilde{m}}(x^0) \left(-\varepsilon \alpha_1 \kappa_1 + \frac{\sqrt{\kappa_2} \|r_\varepsilon\|}{V^{\tilde{m}+\frac{1}{2m_1}}(x^0)} \right) \\ &\quad + \frac{\tilde{m}}{2} (\mu + \kappa_2(\tilde{m} - 1)) V^{\tilde{m}-\frac{1}{m_1}}(\xi) V^{2\tilde{m}+\frac{1}{m_1}}(x^0) \left(\varepsilon \alpha_2 \sqrt{\kappa_2} + \frac{\|r_\varepsilon\|}{V^{\tilde{m}+\frac{1}{2m_1}}(x^0)} \right)^2 \\ &\leq W(x^0) + \tilde{m} V^{2\tilde{m}}(x^0) \left(-\varepsilon \alpha_1 \kappa_1 + \frac{\sqrt{\kappa_2} \|r_\varepsilon\|}{V^{\tilde{m}+\frac{1}{2m_1}}(x^0)} + \frac{\tilde{m} \bar{\mu}}{2} \left(\varepsilon \alpha_2 \sqrt{\kappa_2} + \frac{\|r_\varepsilon\|}{V^{\tilde{m}+\frac{1}{2m_1}}(x^0)} \right)^2 \right). \end{aligned}$$

The substitution of $W(x) = V(x)^{\tilde{m}}$ into the obtained estimate completes the proof. \square

A.2 Proof of Lemma 3

The argumentation is analogous to the proof of [22, Lemma 4.1]. Let us introduce the function $w(t) = \|x(t) - x(0)\|$. Differentiating it along the trajectories of system (11) with controls (13), we get

$$\begin{aligned} \frac{d}{dt} w^2(t) &= 2(x(t) - x(0))^T \sum_{\substack{i=1 \\ s=1,2}}^n F_{si}(J(x(t))) u_{si}(t) e_i \\ &\leq 4\nu n w(t) \max_{\substack{1 \leq i \leq n \\ s=1,2}} |F_{si}(J(x(t))) - F_{si}(J(x(0))) + F_{si}(J(x(0)))| \\ &\leq 2\nu w(t) (Lw(t) + M\|x(0) - x^*\|^m), \end{aligned}$$

where ν is defined in (26). Therefore,

$$\dot{w}(t) \leq \nu (Lw(t) + M\|x(0) - x^*\|^m).$$

Solving the corresponding comparison equation with the initial condition $w(0) = 0$, we obtain the statement of Lemma 3. \square

A.3 Proof of Lemma 4

From [22, Proof of Lemma 3.1], the remainder $R(\tau)$ of the Volterra expansion of $x(t)$ can be represented in the following form:

$$\begin{aligned} R_k(\tau) = & \sum_{i,j=1,2} \sum_{l=1}^n \nabla F_{ik}(J(x^0))^T \int_0^\tau \int_0^t u_{ik}(t) u_{jl}(s) \nabla F_{ik}(J(\xi)) e_l^T \Delta x(s) ds dt \\ & + \frac{1}{2} \sum_{i=1,2} \int_0^\tau \Delta x(t)^T \left(\frac{\partial^2 F_{ik}(J(\eta(t)))}{\partial x^2} \right)^T \Delta x(t) u_{ik}(t) dt, \end{aligned}$$

where $\Delta x(t) = x(t) - x^0$, $0 \leq \|\xi(s) - x^0\| \leq \|\Delta x(s)\|$, $0 \leq \|\eta(t) - x^0\| \leq \|\Delta x(t)\|$ for $0 \leq s \leq t \leq \tau$.

Let $m > 0$ (the case $m = 0$ can be studied in a similar way). Under the conditions of the lemma, we may estimate the Euclidean norm of $R(\tau) = (R_1(\tau), \dots, R_n(\tau))^T$ as

$$\|R(\tau)\| \leq L\tilde{L} \|x^0 - x^*\|^{m-1} \nu^2 \int_0^\tau \int_0^t \|x(s) - x^0\| ds dt + \frac{H\nu\sqrt{n}}{2} \int_0^\tau \|x(\tau) - x^0\|^2 d\tau.$$

The application of inequality (27) from Lemma 3 and the triangle inequality together with the Cauchy–Schwarz inequality completes the proof. \square